JOURNAL OF COMPUTATIONAL PHYSICS 25, 194–198 (1977)

Note

Determination of Spherical Bessel's Functions of an Order Larger than the Argument

The ideas developed in Section 6 of Ref. [1] for the study of the behavior of $j_L(\rho)$ and $n_L(\rho)$ at the origin and for $L \gg \rho$ suggest a direct determination of these functions for $L \gg \rho$.

Consider the radial equation

$$\left[\frac{d^2}{d\rho^2} + \left(1 - \frac{L(L+1)}{\rho^2}\right)\right] u_L(\rho) = 0 \tag{1}$$

of which

$$f_L(\rho) = \rho j_L(\rho), \qquad g_L(\rho) = \rho n_L(\rho) \tag{2}$$

are respectively the regular and the irregular solutions. For $L \gg \rho$ define $\alpha_L(\rho)$ and $\beta_L(\rho)$ by the equations

$$f_L(\rho) = \frac{\rho^{L+1}}{(2L+1)!!} \exp[\alpha_L(\rho)], \qquad g_L(\rho) = \frac{(2L+1)!!}{2L+1} \left(\frac{1}{\rho}\right)^L \exp[\beta_L(\rho)] \tag{3}$$

and substitute $f_L(\rho)$ and $g_L(\rho)$ into (1). One has

$$\frac{d^2\alpha_L}{d\rho^2} + \left(\frac{d\alpha_L}{d\rho}\right)^2 + \frac{2(L+1)}{\rho}\frac{d\alpha_L}{d\rho} + 1 = 0, \tag{4.a}$$

$$\frac{d^2\beta_L}{d\rho^2} + \left(\frac{d\beta_L}{d\rho}\right)^2 - \frac{2L}{\rho}\frac{d\beta_L}{d\rho} + 1 = 0. \tag{4.b}$$

Nonlinear differential Eqs. (4) show that $\alpha_L(\rho)$ and $\beta_L(\rho)$ are even functions of ρ. Take, then, the developments

$$\frac{d\alpha_L}{d\rho} = \sum_{n=0}^{\infty} a_n \rho^{2n+1}, \tag{5.a}$$

$$\frac{d\beta_L}{d\rho} = \sum_{n=0}^{\infty} b_n \rho^{2n-1}, \tag{5.b}$$

and substitute (5.a) and (5.b) respectively, into (4.a) and (4.b). One has, by equating to zero the coefficients of the different powers of ρ in (4.a) and (4.b), the recurrence relations for the $\{a_n\}$ and $\{b_n\}$

$$a_0 = -1/(2L+3), \tag{6.a}$$

$$[2(L+n)+3]\, a_n + \sum_{k=0}^{n-1} a_k \cdot a_{n-k-1} = 0, \tag{6.b}$$

ISSN 0021-9991

and

$$b_0 = 1/(2L - 1), \tag{7.a}$$

$$[2(L - n) - 1]\, b_n - \sum_{k=0}^{n-1} b_k b_{n-k-1} = 0. \tag{7.b}$$

Thus, putting the integrating constants equal to zero, one has for $\alpha_L(\rho)$ and $\beta_L(\rho)$,

$$\alpha_L(\rho) = -\frac{\rho^2}{2(2L+3)} - \frac{\rho^4}{4(2L+5)(2L+3)^2} - \frac{\rho^6}{3(2L+7)(2L+5)(2L+3)^3} - \cdots, \tag{8.a}$$

$$\beta_L(\rho) = \frac{\rho^2}{2(2L-1)} + \frac{\rho^4}{4(2L-3)(2L-1)^2} + \frac{\rho^6}{3(2L-5)(2L-3)(2L-1)^3} + \cdots. \tag{8.b}$$

The series (8) give to $j_L(\rho)$ and $n_L(\rho)$ the correct behavior at the origin and for $L \gg \rho$, if (2) and (3) are taken into account (see Ref. [1]).

It is worth noting that the developments (8) for $\alpha_L(\rho)$ and $\beta_L(\rho)$ are series absolutely convergent for given values of ρ and L.

Take the series for $d\alpha_L/d\rho$ and consider a new set of coefficients $\{a_n'\}$ related to the $\{a_n\}$ by the definitions

$$a_n = a_n' \sigma^{n+1}, \qquad n = 0, 1, \ldots, \tag{9}$$

σ being a constant to be specified below. Introducing these relations in (6.b) one finds that the $\{a_n\}$, and the $\{a_n'\}$ are subjected to the same recurrence rules, i.e.,

$$[2(L+n)+3]\, a_n' + \sum_{k=0}^{n-1} a_k' a_{n-k-1}' = 0, \qquad n = 1, 2, \ldots, \tag{6.b'}$$

Now take $a_0' = 1$, so that, according to (6.a) and (9) one has

$$\sigma = -1/(2L+3) \tag{10}$$

and suppose, by hypothesis, that $|a_k'| \leqslant 1$ for $k = 0, 1, \ldots, n-1$. Then, by (6.b') one obtains

$$|a_n'| \leqslant \frac{1}{2(L+n)+3} \sum_{k=0}^{n-1} |a_k'| \cdot |a_{n-k-1}'| \leqslant \frac{n}{2(L+n)+3} < 1,$$

or, in words, if $| a_k' | \leqslant 1$ for any k smaller or equal to $n - 1$, then $| a_n' | \leqslant 1$. As this statement is true for $n = 1$, then it is equally true by induction for any n

$$| a_n' | \leqslant 1, \qquad n = 0, 1, \ldots. \tag{11}$$

Thus, one can write by (9), (10), and (11)

$$| a_n | \leqslant | \sigma |^{n+1} = \left(\frac{1}{2L + 3}\right)^{n+1}, \tag{12}$$

According to the preceding inequalities for the $| a_n |$ one has the majoration rule for the series (5.a)

$$\left| \frac{d\alpha_L}{d\rho} \right| \leqslant \frac{1}{\rho} \sum_{n=0}^{\infty} \left(\frac{\rho^2}{2L + 3}\right)^{n+1}. \tag{13}$$

The geometrical series on the right-hand side of (13) converges if $\rho < (2L + 3)^{1/2}$, so that the series (5.a) converges absolutely for the same values of ρ. In practice, however, one can go beyond the limit $(2L + 3)^{1/2}$, because the coefficients $\{a_n'\}$ vanish quite rapidly with increasing n (Table I). A fortiori, the series for $\alpha_L(\rho)$ also converges absolutely. The same steps may be taken for the study of the convergence of the series $\beta_L(\rho)$ and $d\beta_L/d\rho$, with only one slight modification.

TABLE I

The Column Headed by n Contains the Number of Terms Kept in the Truncated Series for $\alpha_L(\rho)$, i.e., $(\rho^2/2) \sum_{k=0}^{n-1} [a_k \rho^{2k}/(k + 1)]$[a]

L	ρ	$\alpha_L(\rho)$	n	$j_L(\rho)$
10	1	$-0.21758075 \times 10^{-1}$	3	$0.71165526 \times 10^{-10}$
30	5	-0.19902319	4	$0.42827302 \times 10^{-21}$
30	10	-0.80366138	6	$0.25120574 \times 10^{-12}$

[a] The term for $k = n - 1$ is the first in the series satisfying the condition $| a_{n-1} \rho^{2n}/2n | < 10^{-8}$ because $j_L(\rho)$ is calculated with eight exact significant figures.

New coefficients $\{b_n'\}$ may be introduced, defined by the relations

$$b_n = b_n' \sigma^{n+1}, \qquad n = 0, 1, \ldots, \tag{14}$$

and, consequently, the $\{b_n\}$ and $\{b_n'\}$ satisfy the same recurrence rules (7.b).

Suppose now b_0' and σ are taken, respectively, equal to 1 and $1/(2L - 1)$ (see (7.a)). Can it be proved that $| b_n' | \leqslant 1$ for any n by a method of induction analogous to the one used in obtaining the same inequalities for the $\{a_n'\}$?

As it will be seen below, it cannot, since the proof for the $\{a_n'\}$ requires $| n/[2(L + n) + 3]| < 1$ for any n. But the same is not true for the corresponding ratio for the $\{b_n'\}$, i.e., $| n/[2(L - n) - 1]|$ (see (7.b)), which exceeds unity for any n in the interval $\frac{1}{3}(2L - 1) < n < 2L - 1$.

However, b_0' can be put equal to $1 - \epsilon$, where ϵ is a small positive quantity ($0 \leqslant \epsilon < 1$) calculated in such a way that $| b_n' | \leqslant 1$ is true for any n in the interval $0 \leqslant n \leqslant 2L - 2$. Then, by mathematical induction, this statement must be true for any n in the interval $0 \leqslant n < +\infty$ and one has, according to (14)

$$| b_n | \leqslant \left[\frac{1}{(1 - \epsilon)(2L - 1)}\right]^{n+1}, \qquad n = 0, 1, \ldots . \tag{15}$$

Thus, one obtains for the majoration of $| d\beta_L/d\rho |$ the geometrical series

$$\left| \frac{d\beta_L}{d\rho} \right| \leqslant \frac{1}{\rho} \sum_{n=0}^{\infty} \left[\frac{\rho^2}{(1 - \epsilon)(2L - 1)}\right]^{n+1} \tag{16}$$

which converges for any $\rho < [(1 - \epsilon)(2L - 1)]^{1/2}$. The series (5.b) for $d\beta_L/d\rho$ and a fortiori its integral series (8.b) for $\beta_L(\rho)$ are, then, absolutely convergent for such values of ρ. In practice, however, the convergence goes beyond the limit $(2L - 1)^{1/2}$ (Tables II and III).

TABLE II

The Column Headed by n gives the Number of Terms Maintained in the Truncated Series for $\beta_L(\rho)$, i.e., $(\rho^2/2) \sum_{k=0}^{n-1} [b_k\rho^{2k}/(k + 1)]$[a]

L	ρ	$\beta_L(\rho)$	n	$n_L(\rho)$
10	1	$0.26356718 \times 10^{-1}$	3	0.67221501×10^{9}
30	5	0.21266010	4	$0.77607176 \times 10^{19}$
30	10	0.86060835	7	$0.69083186 \times 10^{10}$

[a] The term for $k = n - 1$ is the first in the series that satisfies the condition $| b_{n-1}\rho^{2n}/2n | < 10^{-8}$ because $n_L(\rho)$ is calculated with eight exact significant figures.

TABLE III

The Truncated Series for $d\beta_L/d\rho$ ($= \rho \sum_{k=0}^{n-1} b_k\rho^{2k}$) is Calculated with n Terms, This Number Being Shown in the Column Headed by n[a]

L	ρ	$d\beta_L/d\rho$	n	$\rho/(2L)$
10	1	$0.52795680 \times 10^{-1}$	4	0.050
30	5	$0.85385653 \times 10^{-1}$	4	0.083
30	10	0.17486921	7	0.167

[a] The term for $k = n - 1$ is the first obeying the inequality $| b_{n-1}\rho^{2n-1} | < 10^{-8}$.

The $\{b_n'\}$ also decrease rapidly with increasing n.

A good test for the convergence of the series for $d\beta_L/d\rho$ (and consequently, for $\alpha_L(\rho)$ and $\beta_L(\rho)$) is to compare an approximate solution to $d\beta_L/d\rho$ and the calculated series for the same quantity. Such a solution is easily obtained from differential Eq. (4.b) for $d\beta_L/d\rho$. By neglecting the small terms $d^2\beta_L/d\rho^2$ and $(d\beta_L/d\rho)^2$ and retaining the term with the large coefficient $(2L)/\rho$, one obtains $d\beta_L/d\rho \simeq \rho/(2L)$.

Table III gives numerical values of the series for $d\beta_L/d\rho$ as well as the corresponding values of its approximate solution $\rho/(2L)$. As expected they are close to one another.

Relations (9) between the a_n and the a_n', and (14), between the b_n and b_n', may serve the practical purpose of calculating the series for $\alpha_L(\rho)$ and $\beta_L(\rho)$ as well as their first derivatives.

Take, for instance, (9) and redefine σ as $1/\sigma = \rho^2$. Therefore $a_0' = -\rho^2/(2L+3)$ and one obtains

$$\frac{d\alpha_L}{d\rho} = \frac{1}{\rho} \sum_{n=0}^{\infty} a_n',$$

$$\alpha_L = \sum_{n=0}^{\infty} \frac{a_n'}{2(n+1)},$$

Finally all the calculations were performed in the Coimbra University SIGMA 5 XEROX computer, using a double-precision FORTRAN-IV programme.

References

1. P. DE A. P. MARTINS, *J. Computational Phys.* **25** (1977), 182–193.

RECEIVED: October 6, 1976

PEDRO DE A. P. MARTINS
Departamento de Física
Universidade de Coimbra
Portugal